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GENERALIZATIONS OF DIXON'S THEOREM ON THE SUM OF A $_3F_2$

J. L. LAVOIE, F. GRONDIN, A. K. RATHIE, AND K. ARORA

ABSTRACT. Twenty-three formulas, closely related to Dixon's theorem in the theory of the generalized hypergeometric series, are obtained. Twenty-six limiting cases are also deduced.

1. INTRODUCTION AND MAIN RESULTS

This paper is to Dixon's theorem what [3] is to Watson's theorem, in the theory of the generalized hypergeometric series.

We start with Dixon's classical result [2, p. 13, equation (1)]

$$(1)\qquad {}_3F_2\left(\begin{matrix} a, & b, & c \\ & 1+a-b, & 1+a-c \end{matrix}\middle| 1\right) = \frac{\Gamma(1+\frac{a}{2})\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\,\Gamma(1+\frac{a}{2}-b)\,\Gamma(1+\frac{a}{2}-c)\,\Gamma(1+a-b-c)},$$

where $R(a-2b-2c) > -2$.

As explained in [3], the sums of certain $_3F_2$ are obtained from Dixon's theorem by a systematic exploitation of the relations between contiguous functions given by Rainville [4, p. 80]. The difference between the parameters found in these series and the corresponding elements in Dixon's formula are positive or negative integers, including zero.

In order to publish most of these apparently new results, a very general form had to be artificially constructed. Its special cases are the potentially useful summation formulas that were initially derived.

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This general form is

$$
(2)\quad {}_3F_2\left(\begin{matrix} a, & b, & c \\ & 1+i+a-b, & 1+i+j+a-c \end{matrix}\middle|\, 1\right)
$$

$$
= \frac{2^{-2c+i+j}\,\Gamma(1+i+a-b)\,\Gamma(1+i+j+a-c)\,\Gamma\left(b-\frac{i}{2}-\frac{|i|}{2}\right)\Gamma\left(c-\frac{1}{2}(i+j+|i+j|)\right)}{\Gamma(a-2c+i+j+1)\,\Gamma(a-b-c+i+j+1)\,\Gamma(b)\,\Gamma(c)}
$$

$$
\times\left\{A_{i,j}\,\frac{\Gamma\left(\frac{a}{2}-c+\frac{1}{2}+\left[\frac{i+j+1}{2}\right]\right)\Gamma\left(\frac{a}{2}-b-c+1+i+\left[\frac{j+1}{2}\right]\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+1+\left[\frac{i}{2}\right]\right)}\right.
$$

$$
\left.+B_{i,j}\,\frac{\Gamma\left(\frac{a}{2}-c+1+\left[\frac{i+j}{2}\right]\right)\Gamma\left(\frac{a}{2}-b-c+\frac{3}{2}+i+\left[\frac{j}{2}\right]\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b+\frac{1}{2}+\left[\frac{i+1}{2}\right]\right)}\right\},
$$

where $R(a-2b-2c) > -2-2i-j$.

The coefficients $A_{i,j}$ and $B_{i,j}$ appear in the tables at the end of the paper. As usual, $[x]$ is the greatest integer less than or equal to x, and its modulus is denoted by $|x|$. Here, $i = -3, -2, -1, 0, 1, 2$ and $j = 0, 1, 2, 3$. But, if $f_{i,j}$ is the left-hand side of (2), the natural symmetry

$$
(3)\qquad f_{i,j}(a, b, c) = f_{i+j,-j}(a, c, b)
$$

makes it possible to extend the result to $j = -1, -2, -3$.

This general form contains many remarkable special cases but, as expected, the results become more intricate as $|i|$ and/or $|j|$ increases, $f_{0,0}$ being Dixon's result. Here is a depiction of all the formulas presented here. The dashes represent results that were considered inappropriate for publication.

—	$f_{4,-2}$	—	—			
$f_{3,-3}$	$f_{3,-2}$	$f_{3,-1}$	$f_{3,0}$	—	—	—
$f_{2,-3}$	$f_{2,-2}$	$f_{2,-1}$	$f_{2,0}$	$f_{2,1}$	$f_{2,2}$	—
$f_{1,-3}$	$f_{1,-2}$	$f_{1,-1}$	$f_{1,0}$	$f_{1,1}$	$f_{1,2}$	—
$f_{0,-3}$	$f_{0,-2}$	$f_{0,-1}$	$f_{0,0}$	$f_{0,1}$	$f_{0,2}$	$f_{0,3}$
—	$f_{-1,-2}$	$f_{-1,-1}$	$f_{-1,0}$	$f_{-1,1}$	$f_{-1,2}$	$f_{-1,3}$
—	—	$f_{-2,-1}$	$f_{-2,0}$	$f_{-2,1}$	$f_{-2,2}$	$f_{-2,3}$
—	—	—	$f_{-3,0}$	$f_{-3,1}$	$f_{-3,2}$	$f_{-3,3}$

The sums of the series in the array on the right are special cases of (2), while the left array is constructed from these results with the aid of (3).

For example,

$$f_{-1,-1}(a,c,b) = f_{-2,1}(a,b,c) = {}_3F_2\left(\begin{matrix} a, & b, & c \\ & a-b-1, & a-c \end{matrix}\middle| 1\right)$$
$$= \frac{\Gamma(a-b-1)\,\Gamma(a-c)}{2^{2c+1}\,\Gamma(a-b-c)\,\Gamma(a-2c)}$$
$$\times\left\{(a-b-1)\frac{\Gamma\left(\frac{a}{2}-c+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b-c\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b\right)}\right.$$
$$\left.+(a-b-2c-1)\frac{\Gamma\left(\frac{a}{2}-c\right)\Gamma\left(\frac{a}{2}-b-c-\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b-\frac{1}{2}\right)}\right\},$$

when $R(a-2b-2c) > 1$.

2. Some limiting cases

The expressions for the sums of certain of our ${}_3F_2$ take the form $\frac{0}{0}$ when $i+j \geq m$ and $c = m$, where m is a positive integer. By applying L'Hôspital's rule, we obtain

$$(4)\qquad {}_3F_2\left(\begin{matrix} a, & b, & m \\ & a-b+l-k+m+1, & a+l+1 \end{matrix}\middle| 1\right)$$
$$= C_{k,l}^{(m)} + D_{k,l}^{(m)}\left\{\psi\left(\frac{a}{2}+\frac{1}{2}\right)-\psi\left(\frac{a}{2}\right)\right.$$
$$+(-1)^{m-1}\,\psi\left(\frac{a}{2}-b+\frac{m+1}{2}+\left[\frac{l-k+1}{2}\right]\right)$$
$$\left.+(-1)^{m}\,\psi\left(\frac{a}{2}-b+\frac{m}{2}+1+\left[\frac{l-k}{2}\right]\right)\right\}$$

where $R(a-2b) > k-2l-2$ and $\psi(z) = \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)}$ is the Psi (or Digamma) function. Again, this is a general form constructed to contain the following twenty-six special cases :

$m=1;$	$k=0,$	$l=0,1,2$	$m=2;$	$k=0,$	$l=0,1$
	$k=1,$	$l=0,1,2$		$k=1,$	$l=0,1$
	$k=2,$	$l=0,1,2$		$k=2,$	$l=0,1,2$
	$k=3,$	$l=0,1,2$		$k=3,$	$l=0,1$

$m=3;$	$k=0,$	$l=0$
	$k=1,$	$l=0$
	$k=2,$	$l=0,1$
	$k=3,$	$l=0$

the coefficients $C_{k,l}^{(m)}$ and $D_{k,l}^{(m)}$ appear in the tables at the end of the paper.

For example, $m=1$, $k=2$ and $l=0$ yields

$$
{}_3F_2\left(\begin{matrix} a, & b, & 1 \\ & a-b, & a+1 \end{matrix}\middle| 1\right)
$$
$$
= \frac{a}{a-b} - \frac{ab}{2(a-b)}\left\{\psi\left(\tfrac{a}{2}+\tfrac{1}{2}\right) - \psi\left(\tfrac{a}{2}\right) + \psi\left(\tfrac{a}{2}-b\right) - \psi\left(\tfrac{a}{2}-b+\tfrac{1}{2}\right)\right\},
$$

when $R(a-2b) > 0$.

3. Concluding remarks

Besides Dixon's theorem, which was our starting point, many new sums of a certain class of generalized hypergeometric series of unit argument have been evaluated. Except for special cases, most of these results do not seem to have appeared in print before. In order to keep control on the length of this paper, certain relations obtained through a single or a double limit were not included.

The interested readers should consult two recent papers, [1] and [5], which deal with an important part of our subject and give useful references.

MATHEMATICA is a general system for doing mathematics by computer. Its aid in obtaining and checking the above results is acknowledged.

TABLE FOR $A_{i,j}$

i \ j	0	1	2	3
3	$5a - b^2 + (a+1)^2 - (2a - b + 1)(b + c)$	—	—	—
2	$\frac{1}{2}(a-1)(a-4) - (b^2 - 5a + 1) - (a - b + 1)(b + c)$	$(b-1)(b-2) - (a - b + 1)(a - b - c + 3)$	$\frac{1}{2}(a - c + 2)(a - 2b - c + 5)$ $[(a - c + 2)(a - 2b + 2) - a(c - 3)]$ $-(b-1)(b-2)(c-2)(c-3)$	—
1	1	$-(a - c + 1)$	$a(a-1) + (b + c - 3)(c - 2a - 1)$	—
0	1	-1	$\frac{1}{2}\{(a - b - c + 1)^2 + (c-1)(c-3) - b^2 + a\}$	$3ab + c(a - b - c + 4) - (a+1)(a+2) - (a-1)(b-1)$
-1	1	1	$b + c - 1$	$(c-1)(c-2) - b(a - c + 1)$
-2	$\frac{1}{2}(a-1)(a - 2b - 2) - c(a - b - 1)$	$(a - b - 1)$	$\frac{1}{2}(a-1)(a - 2b - 2c) + b(b + c)$	$(a - b - 1)(c - 1) - b(b + 1)$
-3	$(a-1)(a - 2b - 2c - 4) + bc$	$(a - b - 2)(a - c - 1) - ac$	$(a - b - 1)(a - b - 2c - 2) - bc$	$b(b+1) + (a-1)(a-b) - c(2a - b - 2)$

Table for $B_{i,j}$

3	$-a+3b^2-(a+3)^2$ $+(2a-3b+5)(b+c)$	—	—	—
2	-2	$-[(b-1)(b-2)$ $-(a-b-2c+5)(a-b-c+3)]$	$-2(a-c+2)(a-2b-c+5)$	—
1	-1	$a-2b-c+3$	$-[(a-b-c+2)(a-b-c+3)$ $-(b-1)(b-c+1)]$	—
0	0	1	-2	$(a+2)(a+4)-b(2a+5)$ $-3c(a-b-c+4)+3$
-1	1	1	$-(b-c+1)$	$(c-1)(c-2)+b(a-2b-c+1)$
-2	2	$a-b-2c-1$	2	$b(a-2c+2)$ $-(b-c+1)(a-b-2c+1)$
-3	$(a-2)(a-2b-2c-3)+3bc$	$(a-b-2)(a-2b-2c-3)+bc$	$(a-b-2)(a-b-2c-1)+bc$	$(a-1)(a-2)-3b(a-b-2)$ $-c(2a-3b-4)$
i / j	0	1	2	3

TABLE FOR $C_{k,l}^{(1)}$

2	$\frac{-(a+2)(a-b+3)(a^2-2ab+7a-3b+9)}{2(b-1)(b-3)}$	$\frac{-(a+2)\,\mathbf{P}_1(a,b)}{2(b-1)(b-2)}$	$\frac{-(a+2)(a^2-2ab+5a-3b+4)}{2(a-b+2)}$	$\frac{-(a+2)(a-b)(a^2-2ab+4a-3b+3)}{2(a-b+1)(a-b+2)}$
1	$\frac{-(a+1)(a-b+2)^2}{(b-1)(b-2)}$	$-(a+1)$	$\frac{-(a+1)(a-b)}{(a-b+1)}$	$\frac{(a+1)(a-b-b^2)}{(a-b)(a-b+1)}$
0	0	0	$\frac{a}{a-b}$	$\frac{a}{a-b}$
l / k	0	1	2	3

where $\mathbf{P}_1(a,b)=(b-2)(2ab-7a+3b-5)+a^2(a-3b+7)$.

TABLE FOR $D_{k,l}^{(1)}$

2	$\frac{-(a-b+3)(a-2b+4)(a-2b+5)\,\mathbf{Q}(a)}{4(b-1)(b-2)(b-3)}$	$\frac{(a-2b+3)(a-2b+4)\,\mathbf{Q}(a)}{4(b-1)(b-2)}$	$\frac{-(a-2b+2)(a-2b+3)\,\mathbf{Q}(a)}{4(b-1)(a-b+2)}$	$\frac{(a-2b+1)(a-2b+2)\,\mathbf{Q}(a)}{4(a-b+1)(a-b+2)}$
1	$\frac{a(a+1)(a-b+2)(a-2b+3)}{2(b-1)(b-2)}$	$\frac{-a(a+1)(a-2b+2)}{2(b-1)}$	$\frac{a(a+1)(a-2b+1)}{2(a-b+1)}$	$\frac{-ab(a+1)(a-2b)}{2(a-b)(a-b+1)}$
0	$\frac{-a(a-b+1)}{2(b-1)}$	$\frac{a}{2}$	$\frac{-ab}{2(a-b)}$	$\frac{ab(b+1)}{2(a-b)(a-b-1)}$
l / k	0	1	2	3

where $\mathbf{Q}(a)=a(a+1)(a+2)$.

TABLE FOR $C^{(2)}_{k,l}$

l \ k	0	1	2	3
2	—	—	$\frac{(a+1)(a+2)(a-b+1)(a^2-2ab+4a-b+2)}{2(b-1)(b-2)}$	—
1	$\frac{a(a+1)(a-b+2)(a-b+3)}{(b-1)(b-3)}$	$\frac{a(a+1)(a-b+1)(a-b\div 2)}{(b-1)(b-2)}$	$a(a+1)$	$\frac{a(a+1)(a-b)}{(a-b+1)}$
0	$\frac{a(a-b)(a-b+1)(a-b+2)}{(a-2b+2)(b-1)(b-2)}$	$\frac{a(a-b+1)}{(a-2b+1)}$	$\frac{a(a-b)}{(a-2b)}$	$\frac{a(a+b^2-b-1)}{(a-b)(a-2b-1)}$

TABLE FOR $D^{(2)}_{k,l}$

l \ k	0	1	2	3
2	—	—	$\frac{-a^2(a+1)(a+2)(a-2b+3)^2}{4(b-1)(b-2)}$	—
1	$\frac{a(a+1)(a-b+2)(a-b+3)(a^2-2ab+3a+b-1)}{2(b-1)(b-2)(b-3)}$	$\frac{-a(a+1)(a-b+2)(a^2-2ab+2a+b-1)}{4(b-1)(b-2)}$	$\frac{a(a+1)(a^2-2ab+a+b-1)}{2(b-1)}$	$\frac{-a(a+1)(a^2-2ab+b-1)}{2(a-b+1)}$
0	$\frac{-a(a-1)(a-b+1)(a-b+2)}{2(b-1)(b-2)}$	$\frac{a(a-1)(a-b+1)}{2(b-1)}$	$\frac{-a(a-1)}{2}$	$\frac{ab(a-1)}{2(a-b)}$

TABLE FOR $C_{k,l}^{(3)}$

l \ k	0	1	2	3
1	—	—	$\frac{-a(a+1)(a-b+1)(a-b+2)(a^2-2ab+3b-2)}{2(a-2b+2)(b-1)(b-2)}$	—
0	$\frac{-a(a-b+1)(a-b+2)(a-b+3)\,\mathbf{P}_2(a,b)}{2(b-1)(b-3)(a-2b+2)(a-2b+3)}$	$\frac{-a(a-b+1)(a-b+2)\,\mathbf{P}_3(a,b)}{2(a-2b+1)(a-2b+2)(b-1)(b-2)}$	$\frac{-a(a-b+1)(a^2-2ab-3a+5b)}{2(a-2b)(a-2b+1)}$	$\frac{-a(a-b)(a^2-2ab-4a+5b+3)}{2(a-2b)(a-2b-1)}$

where $\mathbf{P}_2(a,b) = a^2 - 2ab - a + 5b - 3$
and $\mathbf{P}_3(a,b) = (a+1)(a^2 - 2a + 5b) - b(3a^2 - 2ab + 5b) - 2$.

TABLE FOR $D_{k,l}^{(3)}$

l \ k	0	1	2	3
1	—	—	$\frac{a(a-1)^2(a+1)(a-b+2)(a-2b+2)}{4(b-1)(b-2)}$	—
0	$\frac{-a(a-1)(a-2)(a-b+1)(a-b+2)(a-b+3)}{4(b-1)(b-2)(b-3)}$	$\frac{a(a-1)(a-2)(a-b+1)(a-b+2)}{4(b-1)(b-2)}$	$\frac{-a(a-1)(a-2)(a-b+1)}{4(b-1)}$	$\frac{a(a-1)(a-2)}{4}$

Bibliography

1. B. N. Al-Saquabi, S. L. Kalla, and H. M. Srivastava, *A certain family of infinite series associated with digamma functions*, J. Math. Anal. Appl. **159** (1991), 361–372.
2. W. N. Bailey, *Generalized hypergeometric series*, Cambridge Math. Tract No. 32, Cambridge Univ. Press, Cambridge, 1935; reprinted by Stechert-Hafner, New York, 1964.
3. J. L. Lavoie, F. Grondin, and A. K. Rathie, *Generalizations of Watson's theorem on the sum of a* ${}_3F_2$, Indian J. Math. **34**, no. 1 (1992), 23-32.
4. E. D. Rainville, *Special functions*, Macmillan, New York, 1960; reprinted by Chelsea Publ. Co., Bronx, New York, 1971.
5. H. M. Srivastava, *A simple algorithm for the evaluation of a class of generalized hypergeometric series*, Stud. Appl. Math. **86** (1992), 79–86.

(Lavoie and Grondin) Département de Mathématiques et de Statistique, Université Laval, Québec, Canada G1K 7P4

E-mail address, J. L. Lavoie: jllavoie@mat.ulaval.ca

E-mail address, F. Grondin: fg@mat.ulaval.ca

(Rathie and Arora) Department of Mathematics, Dungar Autonomous College, Bikaner–334001, Rajasthan State, India